

A self-dual poset on objects counted by the Catalan numbers and a type-B analogue

Miklós Bóna*

School of Mathematics
Institute for Advanced Study
Princeton, NJ 08540
bona@ias.edu

Rodica Simion

Department of Mathematics
The George Washington University
Washington, DC 20052
simion@gwu.edu

February 1, 2008

Abstract

We introduce two partially ordered sets, P_n^A and P_n^B , of the same cardinalities as the type-A and type-B noncrossing partition lattices. The ground sets of P_n^A and P_n^B are subsets of the symmetric and the hyperoctahedral groups, consisting of permutations which avoid certain patterns. The order relation is given by (strict) containment of the descent sets. In each case, by means of an explicit order-preserving bijection, we show that the poset of restricted permutations is an extension of the refinement order on noncrossing partitions. Several structural properties of these permutation posets follow, including self-duality and the strong Sperner property. We also discuss posets Q_n^A and Q_n^B similarly associated with noncrossing partitions, defined by means of the excedence sets of suitable pattern-avoiding subsets of the symmetric and hyperoctahedral groups.

Key words: restricted permutations, noncrossing partitions, descents, excedences

1 Introduction

There are more than 150 different objects enumerated by Catalan numbers; [25] contains an extensive list of such combinatorial objects and their properties. Two of the most carefully studied ones are

*This paper was written while the author's stay at the Institute was supported by Trustee Ladislaus von Hoffmann, the Arcana Foundation.

noncrossing partitions and permutations avoiding a 3-letter pattern.

A partition π of the set $[n] := \{1, 2, \dots, n\}$, having blocks $\beta_1, \beta_2, \dots, \beta_k$, is called *noncrossing* if there are no four elements $1 \leq a < b < c < d \leq n$ so that $a, c \in \beta_i$ and $b, d \in \beta_j$ for some distinct blocks β_i and β_j . The set of noncrossing partitions of $[n]$ constitutes a lattice under the refinement order (where $\pi < \nu$ if each block of ν is a union of blocks of π). An investigation of structural and enumerative properties of this lattice was initiated by Kreweras [14], and continued by several authors, e.g., [7], [8], [9], [15], [16], [20], [24]. We denote the lattice of noncrossing partitions of $[n]$ as NC_n^A , since it is a subposet (indeed, a sub-meet-semilattice) of the intersection lattice associated with the type-A hyperplane arrangement in \mathbf{R}^n (which consists of the hyperplanes with equations $x_i = x_j$, for $1 \leq i < j \leq n$). For our purposes, recall from [14] that the poset NC_n^A is ranked with rank function $\text{rk}(\pi) = n - \text{bk}(\pi)$ (where $\text{bk}(\pi)$ denotes the number of blocks of the partition π), rank-symmetric and rank-unimodal with rank sizes given by the Narayana numbers $\left(\frac{1}{n} \binom{n}{k} \binom{n}{k+1}\right)_{0 \leq k < n}$. Furthermore, it is self-dual (see [14], [20]) and has the strong Sperner property (see [20]; that is, for every k , the maximum cardinality of the union of k antichains is the sum of the k largest rank-sizes).

A permutation $\sigma = \sigma_1 \sigma_2 \dots \sigma_n$ of $[n]$, or, in what follows, an n -permutation, is called *132-avoiding* if there are no three positions $1 \leq a < b < c \leq n$ so that $\sigma_a < \sigma_c < \sigma_b$. Classes of restricted permutations avoiding other patterns are defined similarly. Such classes of permutations arise naturally in theoretical computer science in connection with sorting problems (e.g., [13], [27]), as well as in the context of combinatorics related to geometry (e.g., the theory of Kazhdan-Lusztig polynomials [4] and Schubert varieties [2]). The investigation of classes of pattern-avoiding permutations from an enumerative and algorithmic point of view includes [1], [3], [5], [6], [17], [19], [28], to name a few.

In Section 2 of this paper we introduce the partially ordered set P_n^A whose elements are the 132-avoiding n -permutations, ordered by $\sigma < \rho$ if $\text{Des}(\sigma) \subset \text{Des}(\rho)$, where Des denotes the descent set of a permutation. One can think of P_n^A as a Boolean algebra of rank $n - 1$ in which each element S is replicated as many times as there are 132-avoiding permutations with S as the descent set. We show that this poset of restricted permutations is an extension of the lattice of noncrossing partitions NC_n^A by exhibiting a natural order-preserving bijection from the dual order $(NC_n^A)^*$ to the poset P_n^A . This yields the fact that P_n^A has the same rank-generating function as NC_n^A (implicit in [21], where the joint distribution of the descent and major index statistics on 132-avoiding permutations is shown to agree with the joint distribution of the block and rb statistic on noncrossing partitions). It then follows that P_n^A is rank-unimodal, rank-symmetric and strongly Sperner. We also prove that P_n^A is itself a self-dual poset.

We also present type-B analogues of these results. These constitute Section 3 of the paper. The notion of a type-B noncrossing partition of $[n]$ is that first considered by Montenegro [15], systematically studied by Reiner [18], and further investigated by Hersh [11]. These authors show that the type-B noncrossing partitions of $[n]$ form a lattice, NC_n^B , which shares naturally a variety of properties of NC_n^A . In particular, NC_n^B is a rank-unimodal, self-dual, strongly Sperner poset. We define a poset P_n^B into which NC_n^B can be embedded via an order-preserving bijection, with properties analogous to those obtained for type-A. The parallel between the type-A and type-B cases includes the fact that the poset P_n^B is defined in terms of pattern-avoiding elements of the hyperoctahedral group (or signed permutations), ordered by containment of the descent set. The relevant pattern restriction is the simultaneous avoidance of the patterns 21 and $\overline{2} \overline{1}$. This class of restricted signed permutations was considered in [22], where B-analogues are proposed for type-A results in [21] concerning combinatorial

statistics for noncrossing partitions and restricted permutations.

In brief, a class of partitions and one of permutations are equinumerous, and further, the count of the partitions by number of blocks agrees with the count of permutations by number of descents. A similar situation arises for certain type-B analogues of these objects. Our results show that these enumerative relations are manifestations of structural relations between partial orders which can be defined naturally on the objects under consideration. We also discuss posets Q_n^A and Q_n^B of restricted permutations and signed permutations ordered by containment of their sets of excedences. The final section of the paper consists of remarks and problems for further investigation.

2 The type-A case

2.1 A bijection and its properties

It is not difficult to find a bijection from the set of noncrossing partitions of $[n]$ onto that of 132-avoiding n -permutations. Here we exhibit and analyze the structure of such a bijection, f , which will serve as the main tool in proving the results of this section. To avoid confusion, integers belonging to a partition will be called *elements*, while integers belonging to a permutation will be called *entries*. An n -permutation will always be written in the one-line notation, $p = p_1 p_2 \cdots p_n$, with $p_i = p(i)$ denoting its i th entry.

Let $\pi \in NC_n^A$. We construct the 132-avoiding permutation $p = f(\pi)$ corresponding to it as follows. Let k be the largest element of π which is in the same block of π as 1. Put the entry n of p in the k th position, i.e., set $p_k = n$. As p is to be 132-avoiding, this implies that the entries larger than $n - k$ are on the left of n in p , and the entries smaller than or equal to $n - k$ are on the right of n . Delete k from π and apply this procedure recursively, with obvious minor adjustments, to the restrictions of π to the sets $\{1, \dots, k - 1\}$ and $\{k + 1, \dots, n\}$, which are also noncrossing partitions. Namely, if j is the largest element in the same block as $k + 1$, we set $p_j = n - k$, so that the restriction π_1 of π to $\{k + 1, k + 2, \dots, n\}$ yields a 132-avoiding permutation of $\{1, 2, \dots, n - k\}$ placed on the right of n in $p = f(\pi)$. Similarly, if in the restriction π_2 of π to the set $\{1, 2, \dots, k - 1\}$ the largest element in the same block as 1 is equal to j , we set $p_j = n - 1$. Thus, recursively, π_2 yields a 132-avoiding permutation which we realize on the set $\{n - k + 1, n - k + 2, \dots, n - 1\}$ and we place it to the left of n in $p = f(\pi)$. In other words, with a slight abuse of notation, $f(\pi)$ is the concatenation of $f(\pi_2)$, n , and $f(\pi_1)$, where $f(\pi_2)$ permutes the set $\{n - k + 1, n - k + 2, \dots, n - 1\}$ and $f(\pi_1)$ permutes the set $[n - k]$.

To see that this is a bijection note that we can recover the maximum of the block containing the element 1 from the position of the entry n in p , and then proceed recursively.

Example 1 If $\pi = (\{1, 4, 6\}, \{2, 3\}, \{5\}, \{7, 8\})$, then $f(\pi) = 64573812$.

Example 2 If $p = (1, 2, \dots, n)$, then $f(p) = 12 \cdots n$.

Example 3 If $p = (\{1\}, \{2\}, \dots, \{n\})$, then $f(p) = n \cdots 21$.

The following definition is widely used in the literature.

Definition 1 Let $p = p_1 p_2 \cdots p_n$ be an n -permutation. We say that $i \in [n - 1]$ is a descent of p if $p_i > p_{i+1}$. The set of all descents of p is called the descent set of p and is denoted $\text{Des}(p)$.

Now we are in a position to define the poset P_n^A of 132-avoiding permutations we want to study.

Definition 2 Let p and q be two 132-avoiding n -permutations. We say that $p < q$ in P_n^A if $\text{Des}(p) \subset \text{Des}(q)$.

Clearly, P_n^A is a poset as inclusion is transitive. The Hasse diagram of P_4^A is shown in Figure 1.

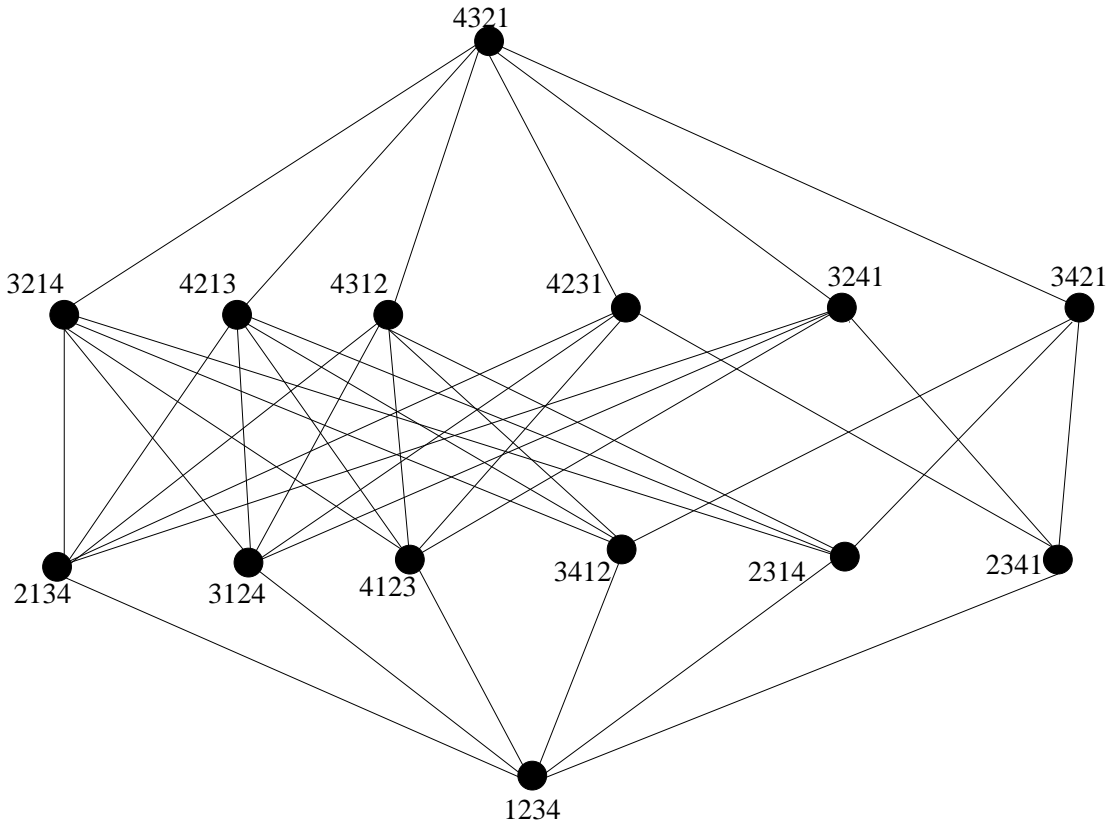


Figure 1: The Hasse diagram of P_4^A

Observation 1 In a 132-avoiding permutation, i is a descent if and only if p_{i+1} is smaller than every entry on its left. Such an element is called a left-to-right minimum. So $p < q$ in P_n^A if and only if the set of positions of left-to-right minima in p is a proper subset of the set of positions of left-to-right minima in q .

The following proposition describes the relation between the blocks of $\pi \in NC_n^A$ and the descent set of the 132-avoiding permutation $f(\pi)$.

Proposition 1 *The bijection f has the following property: Let $i \geq 1$. Then $i \in \text{Des}(f(\pi))$ if and only if $i + 1$ is the smallest element of its block in $\pi \in NC_n^A$.*

Proof: For $n = 1$ and $n = 2$ the statement is clearly true and we use induction on n . Suppose we know the statement for all positive integers smaller than n . Then we distinguish two cases:

1. If 1 and n are in the same block of π , then the construction of $f(\pi)$ starts by putting the entry n in the last slot of $f(\pi)$, then deleting the element n from π . This does not alter either the set of minimum elements of the blocks nor the set of descents. Therefore, this case reduces to the general case for $n - 1$, and is settled by the inductive hypothesis.
2. If the largest element k of the block containing 1 is smaller than n , then as we have seen above, $f(\pi)$ is the concatenation of $f(\pi_2), n, f(\pi_1)$, and $f(\pi_1)$ is not empty. Clearly, by the definition of $f(\pi)$, $k \in \text{Des}(f(\pi))$ and the element $k + 1$ is the minimum of its block. From this and the inductive hypothesis applied to $f(\pi_1)$ and $f(\pi_2)$, the proof follows.

◇

2.2 Properties of P_n^A

Proposition 1 implies that P_n^A is isomorphic to the order on noncrossing partitions in which $\pi < \pi'$ if the set of minima of the blocks of π' is contained in the set of minima of the blocks of π . This yields the first result of this section.

Theorem 1 *The lattice of noncrossing partitions NC_n^A is a subposet of P_n^A .*

Proof: We show that our bijection f is an order-reversing map $NC_n^A \rightarrow P_n^A$. The conclusion then follows from the self-duality of the lattice of noncrossing partitions. Suppose $\pi < \tau$ in NC_n^A . This means π is a finer partition than τ , so every element which is the minimum of its block in τ is also the minimum of its block in π . By Proposition 1 this implies $\text{Des}(f(\tau)) \subset \text{Des}(f(\pi))$, so $f(\pi) > f(\tau)$ in P_n^A . ◇

Clearly, P_n^A is a ranked poset (with rank function $\text{rk}_{P_n^A}(p) = \#\text{Des}(p)$), and we have $\text{rk}_{NC_n^A}(\pi) = n - 1 - \text{rk}_{P_n^A}(f(\pi))$.

Corollary 1 *The poset P_n^A is rank-symmetric, rank-unimodal and strongly Sperner, and its rank generating function is equal to that of NC_n^A .*

Proof: The properties of the rank sizes of P_n^A are immediate consequences of Proposition 1 and the corresponding properties known to hold for NC_n^A . Moreover, every antichain of P_n^A is, via the bijection f , an antichain of NC_n^A , and the strong Sperner property of P_n^A follows from the strong Sperner property of NC_n^A . \diamond

We now turn to showing that P_n^A is self-dual, based on the next lemma.

For $S \subseteq [n-1]$, let $Perm_n(S)$ denote the number of 132-avoiding n -permutations with descent set S .

Lemma 1 *Let S be any subset of $[n-1]$ and let $\alpha(S)$ denote its “reverse complement,” that is, $i \in \alpha(S) \iff n-i \notin S$. Then $Perm_n(S) = Perm_n(\alpha(S))$.*

Proof: We use induction on n . For $n = 1, 2, 3$ the statement is true. Now suppose we know it for all positive integers smaller than n . Denote by t the smallest element of S , and let p be a 132-avoiding n -permutation whose descent set is S .

1. Suppose that $t > 1$. Then we have $p_1 < p_2 < \dots < p_t$ and, because p avoids the pattern 132, the values of p_1, p_2, \dots, p_t are *consecutive* integers. So, for given values of p_1 and t , we have only one choice for p_2, p_3, \dots, p_t . This implies

$$Perm_n(S) = Perm_{n-(t-1)}(S - (t-1)), \quad (1)$$

where $S - (t-1)$ is the set obtained from S by subtracting $t-1$ from each of its elements.

On the other hand, we have $n-t+1, n-t+2, \dots, n-1 \in \alpha(S)$, meaning that in any permutation q counted by $Perm_n(\alpha(S))$ the chain of inequalities $q_{n-t+1} > q_{n-t+2} > \dots > q_n$ holds. To avoid forming a 132-pattern in q , we must have $(q_{n-t+2}, \dots, q_n) = (t-1, t-2, \dots, 1)$. Therefore,

$$Perm_n(\alpha(S)) = Perm_{n-(t-1)}(\alpha(S)|n-(t-1)) \quad (2)$$

where $\alpha(S)|n-(t-1)$ denotes the set obtained from $\alpha(S)$ by removing its last $t-1$ elements. Clearly, $Perm_{n-(t-1)}(S - (t-1)) = Perm_{n-(t-1)}(\alpha(S)|n-(t-1))$ by the induction hypothesis, so equations (1) and (2) imply $Perm_n(S) = Perm_n(\alpha(S))$.

2. If $t = 1$, but $S \neq [n-1]$, then let u be the smallest index which is *not* in S . Then again, to avoid forming a 132-pattern, the value of p_u must be the smallest positive integer a which is larger than p_{u-1} and is not equal to any p_i for $i \leq u-1$. So again, we have only one choice for p_u . On the other hand, the largest index in $\alpha(S)$ will be $n-u$. Therefore, in permutations q counted by $Perm_n(\alpha(S))$, we must have $q_{n-u} = 1$ as Observation 1 implies that q_{n-u} must be the rightmost left-to-right minimum in such permutations, and that is always the entry 1.

In order to use this information to reduce our permutations in size, we define $S' \subset [n-2]$ as follows: $i \in S'$ if and only if either $i < u$ and then, by the definition of u , $i \in S$, or $i > u$ and $i+1 \in S$. In other words, we decrease elements larger than u by 1; intuitively, we remove u from $[n-1]$, and translate the interval on its right one notch to the left. If we now take $\alpha(S')$, that will consist of entries j so that $j < n-u$ and $(n-1) - (j-1) = n-j \notin S$. So in other

words, we simply remove $n - u$ from $[n - 1]$ (there has been nothing on the right of $n - u$ in $\alpha(S)$ to translate). Note that the size of $\alpha(S)$ decreases with this operation as $n - u \in \alpha(S)$. As we have seen in the previous paragraph, we had only one choice for p_u and p_{n-u} , so removing them this way does not change the number of permutations with a given descent set. Thus we have $\text{Perm}_n(S) = \text{Perm}_{n-1}(S')$, and also $\text{Perm}_n(\alpha(S)) = \text{Perm}_{n-1}(\alpha(S'))$. By induction hypothesis, the right hand sides of these two equations agree, and therefore the left hand sides must agree, too.

Example 4 If $n = 8$ and $S = \{1, 6\}$, and so $\alpha(S) = \{1, 3, 4, 5, 6\}$, then $u = 2$, $n - u = 6$, and indeed, $S' = \{1, 5\}$ and $\alpha(S') = \{1, 3, 4, 5\}$.

3. Finally, if $S = [n - 1]$, then the statement is trivially true as $\text{Perm}_n(S) = \text{Perm}_n(\alpha(S)) = 1$.

So we have seen that $\text{Perm}_n(S) = \text{Perm}_n(\alpha(S))$ in all cases. \diamond

It is now easy to verify that the reverse complementation of the descent set can be used to construct an anti-automorphism of P_n^A .

Theorem 2 *The poset P_n^A is self-dual.*

Proof: It is clear that, in P_n^A , permutations which have the same descent set will cover the same elements and they will be covered by the same elements. The permutations with a prescribed descent set S form an orbit of $\text{Aut}(P_n^A)$ and they can be permuted among themselves arbitrarily by elements of $\text{Aut}(P_n^A)$. On the other hand, Lemma 1 shows that the orbits corresponding to $S \subseteq [n - 1]$ and to its reverse-complement $\alpha(S)$ are equinumerous. Hence, a map $P_n^A \rightarrow P_n^A$ which establishes a bijection between $\{p \in P_n^A : \text{Des}(p) = S\}$ and $\{q \in P_n^A : \text{Des}(q) = \alpha(S)\}$ for each $S \subseteq [n - 1]$ provides an order-reversing bijection on P_n^A . \diamond

2.3 A poset derived from excedences

It is shown in [21] that the joint distribution of the excedence and Denert statistics on 321-avoiding permutations agrees with the joint distribution of the block and rb statistics on noncrossing partitions. This suggests the definition of the poset Q_n^A consisting of the 321-avoiding n -permutations ordered by containment of the set of excedences, and invites the question of how Q_n^A compares with the poset P_n^A .

A permutation σ has an *excedence* at i if $\sigma(i) > i$. For example, the *excedence set* of $\sigma = 32514$ is $\text{Exc}(\sigma) = \{1, 3\}$. Let $\text{exc}(\sigma)$ denote the number of excedences of σ . Following [21], there is a bijection θ from NC_n^A to 321-avoiding n -permutations such that $\text{exc}(\theta(\pi)) = \text{bk}(\pi) - 1$. Namely, if the set of minima of the blocks of $\pi \in NC_n^A$, omitting the block containing 1, is $\{f_2 < \dots < f_k\}$ and the set of maxima of the blocks, again, omitting the block containing 1, is $\{l_2 < \dots < l_k\}$, then let $\theta(\pi)$ be the permutation whose value at $f_i - 1$ is l_i for $i = 2, 3, \dots, k$, and whose

other values constitute an increasing subsequence in the remaining positions. For instance, if $\pi = \{1, 5, 7\}\{2\}\{3, 4\}\{6\}\{8, 10\}\{9\} \in NC_{10}^A$, then we have $(f_2, \dots, f_6) = (2, 3, 6, 8, 9)$ and $(l_2, \dots, l_6) = (2, 4, 6, 9, 10)$, and we obtain $\theta(\pi) = 2\ 4\ 1\ 3\ 6\ 5\ 9\ 10\ 7\ 8$.

Recall from [21] that the set of excedences of $\theta(\pi)$ is precisely $\{f_2 - 1, f_3 - 1, \dots, f_k - 1\}$. Similarly to the case of descents discussed for 132-avoiding permutations, a covering relation $\pi < \pi'$ in NC_n^A corresponds to the deletion of an excedence: $\text{Exc}(\theta(\pi')) = \text{Exc}(\theta(\pi)) - \{i\}$, for a suitable $i \in \text{Exc}(\theta(\pi))$. Hence, taking advantage of the self-duality of NC_n^A , one can establish directly that the poset Q_n^A enjoys the same properties as P_n^A : There is an embedding of NC_n^A into the poset Q_n^A of 321-avoiding n -permutations ordered by containment of the set of excedences; the embedding is rank-preserving and Q_n^A is a strongly Sperner poset.

The fact that the posets P_n^A and Q_n^A have strongly similar properties is not accidental.

Proposition 2 *The posets P_n^A and Q_n^A are isomorphic.*

Proof: For each $S \subseteq [n - 1]$, let $E_n^{321}(S)$ be the set of 321-avoiding n -permutations with excedence set $S \subseteq [n - 1]$.

Let also $D_n^{132}(\alpha(S))$ be the set of 132-avoiding n -permutations with descent set equal to $\alpha(S)$, the reverse-complement of S . Thus, in the notation of the previous subsection, the cardinality of $D_n^{132}(\alpha(S))$ is $\text{Perm}_n(\alpha(S))$.

We construct a bijection $s: E_n^{321}(S) \rightarrow D_n^{132}(\alpha(S))$ (illustrated by example 5). If $p \in E_n^{321}(S)$, then, as seen earlier in the definition of θ , the entries p_j with $j \notin S$ form an increasing subsequence. This, and the definition of excedence imply that p_j is a *right-to-left minimum* (that is, smaller than all entries on its right) if and only if $j \notin \text{Exc}(p) = S$.

Now let $p' = p_n p_{n-1} \cdots p_1$ be the reverse of p . Then p' is a 123-avoiding permutation having a left-to-right minimum at position $i \leq n$ exactly if $n + 1 - i \notin S$.

There is exactly one 132-avoiding permutation p'' which has this same set of left-to-right minima at these same positions [19]. Namely, p'' is obtained by keeping the left-to-right minima of p' fixed, and successively placing in the remaining positions, from left to right, the smallest available element which does not alter the left-to-right minima. We set $s(p) = p''$. Observation 1 then tells us that $i \in \text{Des}(p'')$ if and only if $n - i \notin S$, in other words, when $i \in \alpha(S)$, and so p'' belongs indeed to $D_n^{132}(\alpha(S))$.

It is easy to see that s is invertible. Clearly, p' can be recovered from p'' as the only 123-avoiding permutation with the same values and positions of its left-to-right minima as p'' . (All entries which are not left-to-right minima are to be written in decreasing order). Then p can be recovered as the reverse of p' .

The bijections $s: E_n^{321}(S) \rightarrow D_n^{132}(\alpha(S))$ for all the choices of $S \subseteq [n - 1]$ produce an order-reversing bijection from Q_n^A to P_n^A . But P_n^A is self-dual, so the proof is complete. \diamond

Example 5 Take $p = 2\ 4\ 1\ 6\ 3\ 5\ 9\ 10\ 7\ 8 \in E_{10}^{321}(S)$ for $S = \{1, 2, 4, 7, 8\}$. Then its reversal $p' = 8\ 7\ 10\ 9\ 5\ 3\ 6\ 1\ 4\ 2$ has left-to-right minima 8, 7, 5, 3, 1 in positions 1, 2, 5, 6, 8. We obtain $s(p) = p'' = 8\ 7\ 9\ 10\ 5\ 3\ 4\ 1\ 2\ 6$, a permutation in $D_{10}^{132}(\{1, 4, 5, 7\})$.

3 The type-B case

3.1 The type-B noncrossing partitions

The hyperplane arrangement of the root system of type B_n consists of the hyperplanes with equations $x_i = \pm x_j$ for $1 \leq i < j \leq n$ and the coordinate hyperplanes $x_i = 0$, for $1 \leq i \leq n$. The subspaces of \mathbf{R}^n arising as intersections of hyperplanes from among these can be encoded by partitions of $\{1, 2, \dots, n, \bar{1}, \bar{2}, \dots, \bar{n}\}$ satisfying the following properties: i) if $\{a_1, \dots, a_k\}$ is a block, then $\{\bar{a}_1, \dots, \bar{a}_k\}$ is also a block, where the bar operation is an involution; and ii) there is at most one block, called *the zero-block*, which is invariant under the bar operation. The collection of such partitions are the *type-B partitions of $[n]$* . If $1, 2, \dots, n, \bar{1}, \bar{2}, \dots, \bar{n}$ are placed around a circle, clockwise in this order, and if cyclically successive elements of the same block are joined by chords drawn inside the circle, then, following [18], the class of *type-B noncrossing partitions*, denoted NC_n^B , is the class of type-B partitions of $[n]$ which admit a circular diagram with no crossing chords. Alternatively, a type-B partition is noncrossing if there are no four elements a, b, c, d in clockwise order around the circle, so that a, c lie in one block and b, d lie in another block of the partition. The total number of type-B noncrossing partitions of $[n]$ is $\binom{2n}{n}$ (see [18]). As in the case of type A, the refinement order on type-B partitions yields a geometric lattice (in fact, isomorphic to a Dowling lattice with an order-2 group), and the noncrossing partitions constitute a sub-meet-semilattice as well as a lattice in its own right. As a poset under the refinement order, NC_n^B is ranked, with $\text{rk}(\pi) = n - \#(\text{ of pairs of non-zero blocks })$. For example, $\pi = \{1, \bar{3}, \bar{5}\}, \{\bar{1}, 3, 5\}, \{4\}, \{\bar{4}\}, \{2, \bar{2}\}$ is an element of NC_5^B having 2 pairs of non-zero blocks and its rank is equal to 3. The rank-sizes in NC_n^B are given by $\binom{\binom{n}{k}}{k}_{0 \leq k \leq n}$ (see [18]).

The numerous properties of NC_n^A which also hold for NC_n^B (as shown in [18], [11]), establish the latter as a natural B-analogue. In particular, NC_n^B is a self-dual, rank-unimodal, strongly Sperner poset, analogously to the properties of NC_n^A of concern in Section 2. We now turn to a type-B counterpart of the restricted permutations considered in the preceding section.

3.2 A class of pattern-avoiding signed permutations

We will view the elements of the hyperoctahedral group B_n as signed permutations written as words of the form $b = b_1 b_2 \dots b_n$ in which each of the symbols $1, 2, \dots, n$ appears, and may or may not be barred. Thus, the cardinality of B_n is $n!2^n$. To find a B-analogue of the poset P_n^A , we need a subset of B_n whose cardinality is $\#NC_n^B = \binom{2n}{n}$, which is characterized via pattern-avoidance, and over which the distribution of the descent statistic agrees with the distribution across ranks of the type-B noncrossing partitions of $[n]$. Such a class of signed permutations is $B_n(12, \bar{2}\ \bar{1})$ which appears in [22]. We include its description for the reader's convenience.

Consider the elements of B_n which avoid simultaneously the patterns 21 and $\bar{2} \bar{1}$. That is, the set of elements $b = b_1 b_2 \cdots b_n \in B_n$ such that there are no indices $1 \leq i < j \leq n$ for which i) either both b_i, b_j are barred, or neither is barred, and ii) $|b_i| > |b_j|$ (the absolute value of a symbol means $|a| = a$ if a is not barred, and $|a| = \bar{a}$ if a is barred; effectively, the absolute value removes the bar from a barred symbol). The following is immediate: a $(21, \bar{2} \bar{1})$ -avoiding permutation in B_n is a shuffle of an increasingly ordered subset L of $[n]$ whose elements we then bar, with its increasingly ordered complement in $[n]$. For example, $b = \bar{2}1\bar{3}\bar{5}4\bar{6}7$ is one of $\binom{7}{3}$ elements of $B_7(21, \bar{2} \bar{1})$ associated with the subset $L = \{2, 5, 6\} \subseteq [7]$. Obviously, summing over the choices of L of cardinality ranging from zero to n and over the shuffles, it follows that

$$\#B_n(21, \bar{2} \bar{1}) = \sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n} = \#NC_n^B, \quad (3)$$

as desired.

Furthermore, the distribution of descents over $B_n(21, \bar{2} \bar{1})$ is as desired. We say that $b = b_1 b_2 \dots b_n \in B_n$ has a *descent* at i , for $1 \leq i \leq n-1$, if $b_i > b_{i+1}$ with respect to the total ordering $1 < 2 < \dots < n < \bar{n} < \dots < \bar{2} < \bar{1}$, and that it has a descent at n if b_n is barred. As usual, the *descent set* of b , denoted $\text{Des}(b)$, is the set of all $i \in [n]$ such that b has a descent at i . For example, for $b = \bar{2}1\bar{3}\bar{5}4\bar{6}7$ we have $\text{Des}(b) = \{1, 3, 4, 7\}$. It is then transparent that if $b \in B_n(21, \bar{2} \bar{1})$, then its descent set is precisely the set of positions occupied by barred symbols. In conclusion,

Observation 2 *For an element b of the hyperoctahedral group B_n , let $L(b)$ denote the set of symbols which are barred in b , and $\text{Des}(b)$ denote the descent set of b . Then the map $b \mapsto (L(b), \text{Des}(b))$ gives a bijection between the class of restricted signed permutations $B_n(21, \bar{2} \bar{1})$ and ordered pairs of subsets of $[n]$ of equal cardinality.*

3.3 The poset P_n^B

As the B-analogue of the poset of 132-avoiding permutations P_n^A of the preceding section, we consider the poset P_n^B consisting of the $(21, \bar{2} \bar{1})$ -avoiding elements of the hyperoctahedral group B_n , with the order relation given by $b < b'$ if and only if $\text{Des}(b) \subset \text{Des}(b')$.

Based on the preceding discussion and an encoding of type-B noncrossing partitions appearing in [18], one readily obtains the properties of P_n^B which parallel those of P_n^A .

Theorem 3 *The poset P_n^B of $(12, \bar{2} \bar{1})$ -avoiding elements of the hyperoctahedral group B_n , ordered by containment of the descent set, is an extension of the refinement order on the type-B noncrossing partition lattice NC_n^B . The poset P_n^B has the same rank-generating-function as NC_n^B , therefore it is rank-symmetric and rank-unimodal, and it is a self-dual and strongly Sperner poset.*

Proof: It is immediate from its definition and Observation 2 that P_n^B is a ranked poset (namely, $\text{rk}(b) = \#\text{Des}(b)$) and has rank-sizes given by $\left(\binom{n}{k}^2\right)_{0 \leq k \leq n}$, equal to the rank-sizes in NC_n^B . Also, P_n^B

is a self-dual poset: clearly, if b' is the $(21, \overline{2} \overline{1})$ -avoiding signed permutation which corresponds to the pair $([n] - L(b), [n] - \text{Des}(b))$, then the mapping $b \leftrightarrow b'$ is an order-reversing involution on P_n^B .

Toward checking that there is an order-preserving bijection from NC_n^B to P_n^B , we first recall a fact from [18]: every partition $\pi \in NC_n^B$ can be encoded by a pair $(L(\pi), R(\pi))$ of subsets of $[n]$ whose cardinality is the number of pairs of non-zero blocks of π . Informally, these sets consist of the Left and Right delimiters of non-zero blocks when the elements are read in clockwise order (in the circular diagram of π). More precisely, if $n = 0$ or if π has only a zero-block, we set $L = R = \emptyset$. Otherwise, $\pi \in NC_n^B$ has some non-zero block consisting of cyclically consecutive elements in its diagram. If such a block consists of j_1, j_2, \dots, j_k in clockwise order, then $|j_1|$ belongs to $L(\pi)$ and $|j_k|$ belongs to $R(\pi)$. By deleting this block and its image under barring, a type-B noncrossing partition of $[n - k]$ is obtained and the construction of the sets $L(\pi)$ and $R(\pi)$ is completed by repeating this process as long as non-zero blocks arise. For instance, if $\pi = \{1, \overline{3}, \overline{5}\}, \{\overline{1}, 3, 5\}, \{4\}, \{\overline{4}\}, \{2, \overline{2}\}$, then $L(\pi) = \{3, 4\}$ and $R(\pi) = \{1, 4\}$.

Now suppose that $\pi < \pi'$ in NC_n^B , and that this is a covering relation (i.e., $\text{rk}(\pi') = \text{rk}(\pi) + 1$). Then there exist $l \in L(\pi)$ and $r \in R(\pi)$ such that $L(\pi') = L(\pi) - \{l\}$ and $R(\pi') = R(\pi) - \{r\}$, as a result of the merging of blocks entailed by the covering relation. Thus it is clear that if $\pi \in NC_n^B$ is mapped to the signed permutation $b \in P_n^B$ with the property that $(L(b), \text{Des}(b)) = (L(\pi), R(\pi))$, then one obtains an order-reversing embedding of NC_n^B into P_n^B . Combining this with the self-duality of P_n^B we obtain the desired embedding of NC_n^B into P_n^B .

Finally, the strong Sperner property of P_n^B follows as in type A, from the strong Sperner property of NC_n^B (see [18]) and the rank-preserving embedding of NC_n^B into P_n^B . \diamond

3.4 A poset based on type-B excedences

As in the type-A case, there is a self-dual poset of $\#NC_n^B$ restricted signed permutations ordered by containment of the set of excedences. In fact, there is more than one definition of the excedence statistic in the literature, in the case of the hyperoctahedral group. We briefly mention two possibilities considered in [26].

Given a signed permutation b , let k be the number of symbols which are *not* barred in b . We associate to b an $(n + 1)$ -permutation $\sigma(b)$ by setting $\sigma(b)_{n+1} = k + 1$ and, for $1 \leq i \leq n$, letting $\sigma(b)_i = j$ if b_i is the j th smallest among the symbols $b_1, b_2, \dots, b_n, n + 1$ with respect to the linear ordering $1 < 2 < \dots < n < n + 1 < \overline{1} < \overline{2} < \dots < \overline{n}$. For example, if $b = 1 \overline{3} 2 4 5 \overline{6} \overline{8} 7$, then $\sigma(b) = 1 7 2 3 4 8 9 5 6$. Now, the excedence set of b is defined to be that of $\sigma(b)$. It turns out [10] that for $b \in B_n(21, \overline{2} \overline{1})$ this definition makes the excedence set coincide with the descent set for each b . Therefore, this leads to the poset P_n^B again.

An alternative definition for excedences of “indexed permutations” appears in [26]. Specialized to the hyperoctahedral group it is the following.

Definition 3 *If $b \in B_n$, its excedence set is the union of the sets $S(b)$ and $F(b)$, where $S(b)$ is*

the set of excedences computed in the symmetric group for the permutation $|b_1||b_2|\dots|b_n|$ obtained by removing all bars from the symbols in b , and $F(b) = \{i : b_i = \bar{i}\}$, the set of barred fixed points of b .

Thus, for $b = 1 \bar{3} 2 4 5 \bar{6} \bar{8} 7$ we obtain excedences at $\{2, 6, 7\}$ by either of the two definitions. But $b = \bar{1}3\bar{2}$ has excedences at $\{1, 3\}$ by the first definition (based on $\sigma(b) = 3 1 4 2$), and $\{1, 2\}$ if the second definition is adopted.

For the remainder of this section, we work with the notion of excedence as in Definition 3.

Proposition 3 *Let Q_n^B denote the poset of $(21, \bar{2} \bar{1})$ -avoiding signed permutations in B_n , ordered by containment of their excedence set. The poset Q_n^B is self-dual.*

Proof: Let $b \in B_n$ and b' be the reverse of b . Let b'' be the “barred complement” of b' , that is, $|b''_i| = n + 1 - |b'_i|$, and b''_i is barred if and only if b'_i is not barred.

Then it is straightforward to verify that $i \in S(p'') \cup F(p'')$ if and only if $i \notin S(p) \cup F(p)$. Therefore, the reverse complement operation reverses the inclusion of excedence sets for signed permutations. (Thus, the entire hyperoctahedral group B_n ordered by containment of the excedence set is a self-dual poset.) But, clearly, this involution preserves the $(21, \bar{2} \bar{1})$ -avoidance property, and thus Q_n^B is self-dual. \diamond

By [26], the rank generating function of Q_n^B is equal to that of P_n^B . Therefore it is natural to ask whether the posets P_n^B and Q_n^B are isomorphic, just as their type-A counterparts are (Proposition 2). The answer in this case is negative. Indeed, if $n = 3$ it is straightforward to verify that all atoms of P_3^B are covered by six elements, while the atom $\bar{1} 2 3$ of Q_3^B is covered by seven elements (namely, $\bar{1} \bar{2} 3, \bar{1} 2 \bar{3}, \bar{1} 3 \bar{2}, 2 3 \bar{1}, \bar{2} \bar{3} 1, 2 \bar{1} \bar{3}$, and $\bar{2} 1 \bar{3}$).

4 Remarks and questions for further investigation

1. *Is NC_n^B a subposet of Q_n^B ?* We do not know whether the lattice of type-B noncrossing partitions can be embedded in the poset Q_n^B of $(21, \bar{2} \bar{1})$ -avoiding signed permutations ordered by their excedence set of definition 3.
2. *Self-duality of NC_n^A and NC_n^B extending to self-duality for P_n^A and P_n^B .*

We have seen that *each* of the posets NC_n^A and P_n^A is self-dual and that NC_n^A is a subposet of P_n^A . The same is true for the pair NC_n^B, P_n^B . Both for type A and for type B one can exhibit an order-reversing involution on the larger poset which restricts to an order-reversing involution on the smaller one.

We first construct such an involution g for NC_n^A which will be similar, though not identical, to the involution defined in [20].

Write the elements $1, 2, \dots, n$ clockwise around a circle, and write elements $1', 2', \dots, n'$ interlaced in counterclockwise order, so that $1'$ is between 1 and n , $2'$ is between n and $n - 1$, and so on, i'

is between $n+2-i$ and $n+1-i$. For $\pi \in NC_n^A$, join by chords – as usual – cyclically successive (unprimed) elements belonging to the same block of π . Then define $g(\pi)$ to be the coarsest noncrossing partition on the elements $1', 2', \dots, n'$ so that the chords joining primed elements of the same block do not intersect the chords of π . See Figure 2 for an example.

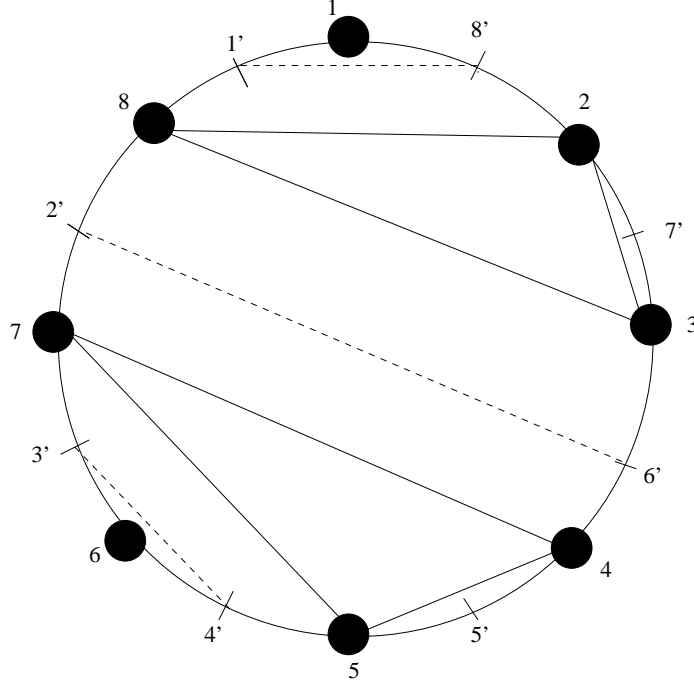


Figure 2: The partition $\pi = (\{1\}, \{2, 3, 8\}, \{4, 5, 7\}, \{6\})$ and its image $g(\pi)$

The map g is certainly a bijection, and it is order-reversing in NC_n^A since merging two blocks of π subdivides a block of $g(\pi)$. We claim that g is also order-reversing on P_n^A . To see this, observe that for any $i > 1$, the element i is the smallest in its block in π if and only if the element $(n+2-i)'$ is *not* the smallest in its block in $g(\pi)$. Indeed, the definition of g implies that exactly one of i and $(n+2-i)'$ can be connected to smaller elements by a chord. Therefore, g takes the set of block-minima (not equal to 1) of π into its reverse complement in $[n+2]$, so g is indeed order-reversing on P_n^A .

In the type- B case, one can obtain an analogous bijection h in a similar way: take the circular (clockwise) representation of π , then write the elements $1', 2', \dots, n', \bar{1}', \bar{2}', \dots, \bar{n}'$ so that the primed numbers interlace the unprimed, placing $1'$ between 1 and \bar{n} and continuing counter-clockwise. For $\pi \in NC_n^B$, define $h(\pi)$ as above, that is, as the unique coarsest partition on the primed set whose chords do not cross those of π . Then h is certainly an order-reversing bijection of NC_n^B , and as above, it reverses the containment of the sets $L(\pi)$ and $R(\pi)$, so it does extend to an order-reversing bijection of P_n^B .

3. The Möbius function and order complexes of P_n^A and P_n^B .

It is easy to write an expression for the number $c_m(P_n^B)$ of chains $\hat{0} < b^1 < b^2 < \dots < b^m < \hat{1}$ of

length $m + 1$ in P_n^B , for $m \geq 0$. Of course, $c_0(P_n^B) = 1$, and

$$c_m(P_n^B) = \sum_{0 < k_1 < k_2 < \dots < k_m < n} \binom{n}{k_1} \binom{n}{k_2} \dots \binom{n}{k_m} \frac{n!}{k_1!(k_2 - k_1)! \dots (k_m - k_{m-1})!(n - k_m)!} \quad (4)$$

since under the correspondence $b^i \leftrightarrow (L(b^i), \text{Des}(b^i))$ a chain in P_n^B corresponds to an m -tuple of subsets $(L(b^i))$ and a chain of subsets $\text{Des}(b^1) \subset \text{Des}(b^2) \subset \dots \subset \text{Des}(b^m)$ of $[n]$, with $\#L(b^i) = \#\text{Des}(b^i) = k_i$. In turn, this leads to an expression for the Möbius function of P_n^B , $\mu_{P_n^B}(\hat{0}, \hat{1}) = \sum_{m \geq 0} (-1)^{m-1} c_m(P_n^B)$.

These expressions can be regarded as partial success with the computation of the zeta polynomial and the Möbius function. It would be interesting to elucidate further the question of these invariants for P_n^A and P_n^B , and to describe the order complexes of these posets.

4. Other posets of combinatorial objects with similar properties.

The behavior of noncrossing partitions and restricted permutations suggests the following question: what other combinatorial objects admit a natural partial order which is self-dual and possibly, has other nice properties? A natural candidate is the class of *two-stack sortable permutations* [29]. It is known [12] that there are as many of them with k descents as with $n - 1 - k$ descents. However, the poset obtained by the descent ordering is not self-dual, even for $n = 4$, so another ordering is needed.

Similarly, the type-D noncrossing partitions and the interpolating BD-noncrossing partitions do not, in general, form self-dual posets when ordered by refinement (see [18]). However, it may be interesting to find corresponding classes of pattern-avoiding elements in the Weyl group for type D, along with an order-preserving embedding $NC_n^D \rightarrow P_n^D$ analogous to the type-A and B cases.

References

- [1] E. Barcucci, A. Del Lungo, and E. Pergola, Permutations with one forbidden subsequence of increasing length, Extended Abstracts, Proc. 9th Conf. Formal Power Series and Algebr. Combin. (Vienna), 1997.
- [2] S.C. Billey, Pattern avoidance and rational smoothness of Schubert varieties, Adv. in Math. **139** (1998) 141-156.
- [3] M. Bóna, Exact enumeration of 1342-avoiding permutations; A close link with labeled trees and planar maps Journal of Combinatorial Theory, Series A, **80** (1997), 257-272.
- [4] F. Brenti, Combinatorial properties of the Kazhdan-Lusztig R -polynomials for S_n , Adv. in Math. **126** (1997) 21-51.
- [5] T. Chow and J. West, Forbidden sequences and Chebyshev polynomials, Discrete Math., to appear.
- [6] S. Dulucq, S. Gire, and J. West, Permutations with forbidden subsequences and nonseparable planar maps, Proc. 5th Conf. Formal Power Series and Algebr. Combin. (Florence, 1993), Discrete Math. **153** (1996) 85-103.
- [7] P. Edelman, Chain enumeration and noncrossing partitions, Discrete Math. **31** (1980) 171-180.
- [8] P. Edelman, Multichains, noncrossing partitions and trees, Discrete Math. **40** (1982) 171-179.

- [9] P. Edelman and R. Simion, Chains in the lattice of noncrossing partitions. *Discrete Math.* **126** (1994), no. 1-3, 107–119.
- [10] J. Galovich, personal communication, February 1999.
- [11] P. Hersh, Deformation of chains via a local symmetric group action, *Electronic J. Combin.*, to appear.
- [12] B. Jacquard and G. Schaeffer, A bijective census of nonseparable planar maps. *J. Combin. Theory Ser. A* **83** (1998), no. 1, 1–20.
- [13] D. Knuth, “The Art of Computer Programming,” vol. 3, Addison-Wesley, Reading, MA, 1973.
- [14] G. Kreweras, Sur les partitions non croisées d’un cycle, *Discrete Math.* **1** (1972), no. 4, 333–350.
- [15] C. Montenegro, The fixed point non-crossing partition lattices, manuscript, 1993.
- [16] A. Nica and R. Speicher, A “Fourier transform” for multiplicative functions on non-crossing partitions, *J. Algebraic Combin.* **6** (1997) 141–160.
- [17] J. Noonan and D. Zeilberger, The enumeration of permutations with a prescribed number of “forbidden” patterns, *Adv. in Appl. Math.* **17** (1996) 381–407.
- [18] V. Reiner, Non-crossing partitions for classical reflection groups. *Discrete Math.* **177** (1997), no. 1-3, 195–222.
- [19] R. Simion, F. W. Schmidt, Restricted permutations, *European Journal of Combinatorics*, **6** (1985), 383–406.
- [20] R. Simion, D. Ullman, On the structure of the lattice of noncrossing partitions, *Discrete Math.* **98** (1991), no. 3, 193–206.
- [21] R. Simion, Combinatorial statistics on noncrossing partitions, *J. Combin. Theory Ser. A* **66** (1994) 270–301.
- [22] R. Simion, Type-B analogues of combinatorial statistics on noncrossing partitions and restricted permutations, manuscript in preparation.
- [23] R. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry. Graph theory and its applications: East and West (Jinan, 1986), 500–535, *Ann. New York Acad. Sci.*, 576, New York Acad. Sci., New York, 1989.
- [24] R. Stanley, Parking functions and noncrossing partitions, *Electronic J. Combin.* **4** (1997) R20, 14pp.
- [25] R. Stanley, “Enumerative Combinatorics,” vol. 2, Cambridge University Press, New York/Cambridge, 1999.
- [26] E. Steingrímsson, Permutation statistics of indexed permutations, *European J. Combin.* **15** (1994) 187–205.
- [27] R. Tarjan, Sorting using networks of queues and stacks, *J. Assoc. Comput. Mach.* **19** (1972) 341–346.
- [28] J. West, Generating trees and forbidden subsequences, *Proc. 6th Conf. Formal Power Series and Algebr. Combin.* (New Brunswick, NJ, 1994), *Discrete Math.* **157** (1996) 363–374.
- [29] D. Zeilberger, A proof of Julian West’s conjecture that the number of two-stack-sortable permutations of length n is $2(3n)!/((n+1)!(2n+1)!)$, *Discrete Math.* **102** (1992), no. 1, 85–93.